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ON DISCONTINUITY SURFACES IN A MEDIUM DEVOID OF "PROPER" PRESSURE

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A new type of discontinuity surfaces which it is necessary to use in certain models of media devoid of proper pressure is considered.

Models of media devoid of proper pressure and containing a new type of discontinuity surface are fairly widely used for defining various flows such as dispersed components of multiphase mixtures, investigated in the approximation of multi-fluid continuous medium, motion of liquid drops in regions of disintegration ("splitting off") of the liquid continuity that occur in fast processes in pulsed type equipment, collision of hypersonic streams in the limit case of infinite compression (increase of density) at shocks, etc. The main feature distinguishing such discontinuities from shock waves and contact (tangential) discontinuities in classical gasdynamics is associated with finite surface (or even linear) densities at these. Owing to this the discontinuity has its proper mass, momentum, and energy whose variation is in particular due to the precipitation from it of matter of the same phase. Similarly to surface charges and currents in electrodynamics and magnetogasdynamics of perfectly conducting media, the finite surface and linear densities and the related to these infinite volume densities are, evidently, the result of schematization introduced in the construction of the model of a medium.

1. Let us, first, look at what happens to the equations defining the flow of a perfect (inviscid and non-heat-conducting) medium when pressure $p \equiv 0$ and density $\rho \not\equiv 0$. The input system of differential equations of a perfect medium contains grad p in the equation of motion and p / ρ in the formula $i = e + p / \rho$ for specific enthalpy (e is the specific internal energy). Thus, for example, if pressure is disregarded in the equations of one-dimensional unsteady flow, the equation of motion becomes

$$\partial u / \partial t + u \partial u / \partial x = 0 \tag{1.1}$$

where t is the time, x is a space coordinate, and u is the x-component of the velocity vector. Equation (1.1) is usually used for illustrating the properties of discontinuous (generalized) solutions. It is shown (see, e.g., [1]) that the properties of discontinuities (particularly their propagation velocity D) depend on the relation of the integral equation taken as the input one to (1.1). Thus, if the integral equation

$$\oint_{\gamma} u dx - \frac{u^2}{2} dt = 0 \tag{1.2}$$

where γ is an arbitrary closed contour in the xt-plane, is taken as the input equation, then

$$D = (u_+ + u_-) / 2 \tag{1.3}$$

where, and in what follows, the subscript plus (minus) denotes parameters to the right (left) of the discontinuity. A discontinuity of this type obtains only when $u_- > u_+$ and when it moves at higher absolute velocity than the particles ahead of it but lower than those behind it. This situation is shown Fig. 1, a and b in which the discontinuity (the double line) propagates, respectively, to the right ($D > 0$) or to the left ($D < 0$). The separate continuous lines represent the particle trajectories $dx / dt = u$

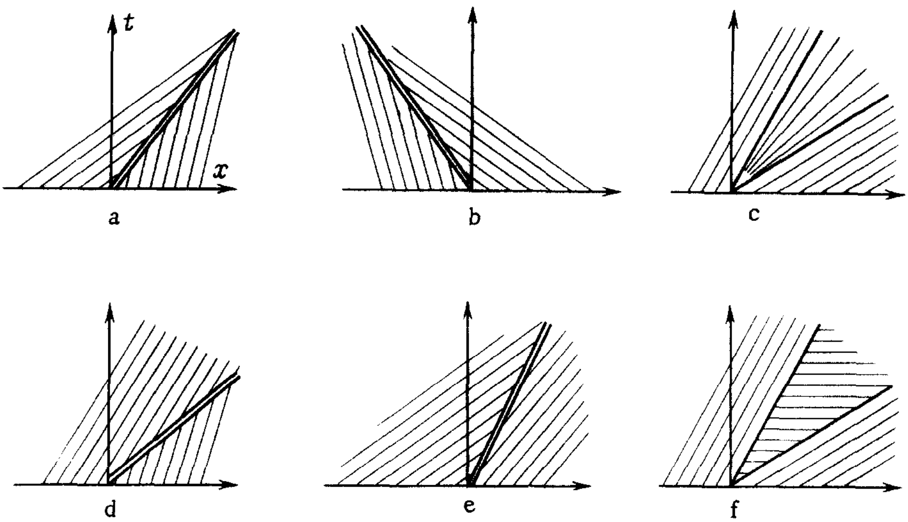

Fig. 1

which are characteristics of Eq. (1.1). When $u_- < u_+$ instead of a discontinuity a centered rarefaction wave is formed (Fig. 1, c) in which u assumes all values contained between u_- and u_+.

Actually there is no arbitrariness in the selection of the differential equation (or equations), since the input laws are defined in integral, not differential form. More exactly, even when such arbitrariness exists, the various integral laws of conservation (e.g., of momentum and moment of momentum) yield the same not different relationships (as well as differential equations) at discontinuities.

Thus when $p \equiv 0$ and $\rho \not\equiv 0$ the system of integral equations of a one-dimensional unsteady flow with plane waves is of the form

$$\oint_\gamma \rho \mathbf{a} (dx - udt) = 0 \tag{1.4}$$

where $\mathbf{u}$ is a vector with components 1, u and $2e + u^2$. System (1.4) relates to the case when the stream is directed along the x-axis. Relations at discontinuities that do not have "surface" density are derived from (1.4) in the usual manner. We have

$$[\rho\,(D-u)]=0,\ \ [\rho\,(D-u)\,u]=0,\ \ [\rho\,(D-u)(2e+u^2)]=0 \quad (1.5)$$
$$([\varphi]=\varphi_+-\varphi_-)$$

It is seen that at this type of discontinuities, as well as on ordinary contact discontinuities $D=u_-=u_+$. Hence generally, as for instance in the problem of disintegration of an arbitrary discontinuity when $u_-\neq u_+$, discontinuities different from (1.5) that carry mass, momentum, and energy are required for constructing the model. The relations that have to be satisfied at such discontinuities are also obtained from (1.4), and are of the form

$$dR\,/\,dt=[\rho\,(D-u)],\quad dRD\,/\,dt=[\rho\,(D-u)\,u] \quad (1.6)$$
$$dR\,(2E+D^2)\,/\,dt=[\rho\,(D-u)(2e+u^2)]$$

where $d\,/\,dt$ is the total derivative with respect to t taken along the trajectory of the discontinuity, R is the surface density, RE is the surface internal energy(energy of unit surface area), and E is the specific internal energy (per unit of mass at the discontinuity.

Generally, according to (1.6) R, D, and E of similar discontinuities are determined by the whole previous history of variation from the instant of formation.The self-similar problem of an arbitrary discontinuity disintegration is in some sense an exception. Let us consider that problem, since its solution is interesting not only in itself. It provides initial values of R, D and E for any discontinuous distributions of parameters, i.e. ρ, u, and e at $t=0$. The dimensional analysis of the considered self-similar problem shows that D and E are constants, and $R=\beta t$ where β is also constant. Form this and (1.6) we have

$$\beta=[\rho\,(D-u)],\quad \beta D=[\rho\,(D-u)\,u] \quad (1.7)$$
$$\beta\,(2E+D^2)=[\rho\,(D-u)(2e+u^2)]$$

Elimination of β from the first two equalities of this system yields a quadriatic equation for D whose root is

$$D=(u_+\alpha_+ + u_-\alpha_-)\,/\,(\alpha_+ + \alpha_-),\quad \alpha=\sqrt{\rho} \quad (1.8)$$

which for $u_- > u_+$ satisfies the inequalities $u_- > D > u_+$ which represent conditions for the particles to reach the discontinuity from both sides.

Values of β and E are then obtained from the first and third of Eqs. (1.7). When $u_+=u_-$ from (1.8) we have $D=u_+=u_-$ and, by virtue of (1.7), also$\beta=0$. Thus in this case (1.7) and (1.8) yield the same results as (1.5). This is obvious, since the discontinuities at which conditions (1.5) are satisfied represent a particular case of discontinuities that satisfy conditions (1.6) when $R=0$, while $\beta=0$ implies that $R=\beta t=0$. In the case of root (1.8) the discontinuity and particle trajectories are the same as in the case of (1.3), i.e. they conform to Fig. 1, a and b . Moreover, when $\rho_+=\rho_-$ (1.8) reduces to (1.3). The second root of the mentioned above quadratic equation is $D=(u_+\alpha_+ - u_-\alpha_-)\,/\,(\alpha_+ - \alpha_-)$ and corresponds to cases represented in Fig. 1, d and e . Unlike in Fig. 1, a and b , here the particles reach the discontinuity not from two sides, but from one side and leave it from the other. Which side is which when $u_- > u_+$ depends only on the relation between ρ_- and ρ_+. Thus Fig. 1, d corresponds to $\rho_+ < \rho_-$ and Fig. 1, e to $\rho_+ > \rho_-$. Although in the absence of some special supplementary mechanisms, the emission of particles by the discontinuity appears physically unreal, it cannot be excluded that in some situations the second root may prove to be useful in spite of the fact that it defines an infinite velocity of the discontinuity when $\rho_+ = \rho_-$.

When $u_+ > u_-$ a new region free of particles ($\rho \equiv 0$) is created by the disintegration of an arbitrary discontinuity in the system defined by Eqs. (1.4). In Fig. 1, f

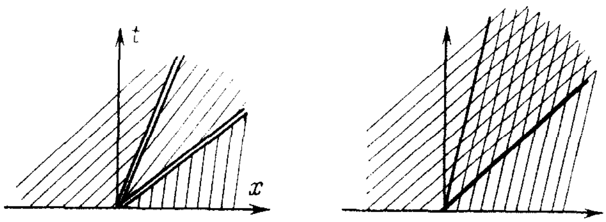

Fig. 2.

that region is shown shaded by horizontal lines. This case was recently considered by G.A. Antonov in the problem of splitting off. When $u_+ > u_-$ it is possible to derive two more solutions with discontinuity (1.7) which correspond to both roots for D. However these solutions, unlike the previous one, are "unstable" with respect to small "smoothing" of the initial discontinuity. The respective reasoning is similar to that used, for instance in [1] in the analysis of a similar situation in the case of (1.1) and (1.2). We would, finally, draw the attention to the difference in the cases represented in Figs. 1, c and f.

2. Let us now consider two comparatively simple problems in which the medium can be considered as being to some extent devoid of proper pressure. First, we shall try to show to what correspond in gasdynamics the investigated above discontinuities which we shall subsequently call "sheet type", or simply "sheet" discontinuities. For this we shall consider the self-similar problem of collison of two uniform streams of gas. Let the relative velocity of the streams be hypersonic, i.e. considerably higher than the speed of sound in each of them. The result of the interaction is the generation of two shock waves which bound the shock or compressed layer, as shown in Fig. 2, a. The gas reaches the compressed layer through both shock waves with the densities ρ^+ and ρ^- of gases entering that layer from different sides being, generally, unequal. These gases are separated by a contact discontinuity (the dashed line in Fig. 2, a). The distribution of pressure and velocity in the compressed layer is unform, i.e. $p^+ = p^- = P$ and $u^+ = u^- = D$. We assume that the relative propagation velocitis of shock waves through the gas upstream of them $(D_\pm - u_\pm)$ is considerably higher in absolute value than the respective speeds of sound $a_\pm$, i.e. $M_\pm \equiv |D_\pm - u_\pm| / a_\pm \gg 1$. Then on the strength of the condition of momentum conservation at the shock it is possible to omit, as shown in [2], the oncoming stream pressures p_+ or p_-. To sum up, the conditions for both shocks assume the form

$$P + \rho^{+} (D_{+} - D)^2 = \rho_{+} (D_{+} - u_{+})^2$$
$$P + \rho^{-} (D_{-} - D)^2 = \rho_{-} (D_{-} - u_{-})^2$$

Eliminating in these equalities P and applying the conditions of conservation of mass

$$\rho^{\pm} (D_{\pm} - D) = \rho_{\pm} (D_{\pm} - u_{\pm}) \tag{2.1}$$

we get rid of ρ^{+} and ρ^{-}, which yields the relationship

$$\rho_{+} (D_{+} - u_{+})^2 - \rho_{+} (D_{+} - u_{+})(D_{+} - D) = \rho_{-} (D_{-} - u_{-})^2 - \rho_{-} (D_{-} - u_{-})(D_{-} - D) \tag{2.2}$$

Let now the compression of gas in the shocks increase as $M_{\pm} \to \infty$, which means that the ratios $\rho^{\pm}/\rho_{\pm}$ also tend to infinity. In the case of perfect gas with adiabatic exponent $\varkappa$ these ratios tend to $\varepsilon^{-1} \equiv (\varkappa + 1) / (\varkappa - 1)$, and this means that $\varkappa \to 1$ and $\varepsilon \to 0$. Then, as implied by the condition of conservation of the energy flux at the shock, an infinite compression requires that $M_{\pm} \gg \varepsilon^{-1/2}$. If $\rho^{\pm}/\rho_{\pm} \to \infty$, then in conformity with (2.1) $D_{\pm} \to D$, which makes it possible to omit the second terms in (2.2). Extracting the square roots of both parts of the obtained equalities we obtain both roots of D in Sect. 1. However, since in our problem the gas penetrates the compressed layer through both shocks, only the first root, i.e. (1.8), has any physical meaning. If we introduce for the shock layer whose thickness for $\varepsilon = 0$ is also zero the surface density R and the specific internal energy E, the equations for the determination of these roots coincide with (1.7) and for nonself-similar problems with (1.6). It is permissible to introduce the surface tension R also when $\varepsilon > 0$. For its determination we obtain the equation

$$\frac{dR}{dt} = \rho_{+} (D_{+} - u_{+}) - \rho_{-} (D_{-} - u_{-}) \equiv [\rho (D - u)] + \frac{\rho_{+}^2}{\rho^{+}} (D_{+} - u_{+}) - \frac{\rho_{-}^2}{\rho^{-}} (D_{-} - u_{-}), \quad R(t) = \int\limits_{x_{-}(t)}^{x_{+}(t)} \rho(x, t)\, dx \tag{2.3}$$

where $x = x_{\pm}(t)$ is the equation of shock wave trajectories and the second form of the right-hand side is obtained from the first with allowance for (2.1). As shown above, the right-hand side of (2.3) tends to $[\rho (D - u)]$ with increasing ratios $\rho^{\pm}/\rho_{\pm}$. Hence for $\varepsilon \neq 0$ its second and third terms define the error of the model of a medium with zero proper pressure.

In the example of "hypersonic" collision of streams just considered pressure may only be disregarded outside the shock layer. Since in the layer $p \equiv P \to \infty$ as $\varepsilon \to 0$, the pressure cannot, obviously, be neglected. However even in the latter case the introduction of sheet type discontinuities yields a picture of flow outside shock layers that is qualitatively (and when $\varepsilon = 0$ also quantitatively) correct. Moreover in many cases the width of these layers, which decreases as $\varepsilon \to 0$, can be small in comparison with the characteristic dimension of the problem.

Let us now consider another problem which also has an exact solution. In the previous problem the gas outside the shock layer had a proper pressure but its contribution to momentum and energy streams was negligibly small. Let now the proper pressure be altogether absent, and the investigated medium be an aggregate of noninteracting particles. In such case with $u_- > u_+$ the problem of discontinuity disintegration contains a "compressed" layer into which particles penetrate from right and left (Fig. 2, b), and the volume density $\rho = \rho_+ + \rho_-$, although exceeding both ρ_+ and ρ_-, does not tend to increase infinitely. In strong rarefaction when the effect of collisions in the layer is immaterial, the flow in it is of the "two-velocity" type. This feature (conversion fom one- to two-velocity model) makes possible to derive a solution without the sheet, as shown in Fig. 2, b. It is, however, possible to introduce also in this case the surface density $R = (\rho_+ + \rho_-)(x_+ - x_-)$, with the nonzero value of R being due to the finiteness of $(x_+ - x_-)$ and not to the infinite increase of volume density. Owing to particle collisons, which always takes place, the stream of particles cannot freely penetrate through one another. This results in the increase of ρ in the compressed layer whose width diminshes. As ρ increases the collision frequency and the proper pressure increase, so that it becomes ultimately necessary to take pressure into account. For example, in the problem of splitting off the latter obviously will take place after $\rho_+ + \rho_-$ exceeds a certain critical value ρ_0 which is a physical characteristic of the medium. After this the interaction of the type shown in Fig. 2, b is transformed in a collision with the formation of a shock layer (Fig. 2, a) in which the medium recovers its "continuity". As far as the author is aware, this feature has not been given due attention, even in the determination of flows with splitting off by the method of Godunov (as shown in [9], solution of the problem of discontinuity disintegration is at the basis of this method).

Finally we point out the rather peculiar character of "convergence" which has to be expected in the numerical integration of system (1.4) by the method of "through" computation. If h is the pitch of the difference grid, its reduction $(h \to 0)$ results in the increase of volume density on the "blurred" sheet in proportion to h^{-1}. In the case of the single-fluid model this should not be harmful, on the contrary, it proves the correctness of obtained results. Outside the sheet the convergence of results is conventional.

3. The most important example of media without proper pressure is the dispersed phase of a mixture of gas and foreign particles. As previously, discontinuities of the sheet type must be introduced here when intersection of trajectories (or streamlines) of particles occurs, in spite of this, the analysis is carried out as before using the two-fluid model (one "fluid" is the gas, the other particles). Let us pass to the invesigation of such discontinuities, bearing in mind that the parameters of gas when the latter passes through the sheet are altered jumpwise owing to the finite action of particles.

Note that such discontinuities were not previously investigated in the analysis of discontinuity surfaces in two-fluid continuous medium (see, e.g. [4, 5]. The first indication of the formation of a sheet appears to have been given in [6], although some concepts to some extent related to similar discontinuities were earlier formualted by A. M. Giliarovskaia and R. E. Sorkin.

Without going into the main assumptions on which the model of a two-fluid continuous medium of the considered type is based, we adduce the integral laws of conservation that will be necessary subsequently. Let Ω be an arbitrary volume independent of t completely filled by at least one of the interacting media, σ its boundary,

$\mathbf{n}$ the unit vector of the outer normal to σ, and $U_n = \mathbf{U}\cdot\mathbf{n}$, where $\mathbf{U}$ is the velocity vector. The parameter of (the second phase) particles will be denoted by subscript s, while those of gas (the carrier phase) will be denoted by the same letters but without subscripts. The integral laws of conservation which define the flow of a mixture of gas and particles in the two-fluid approximation can now be written in the form

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho d\Omega + \iint\limits_{\sigma}\rho U_n d\sigma = 0 \tag{3.1}$$

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho \mathbf{U} d\Omega + \iint\limits_{\sigma}(\rho U_n \mathbf{U} + p\mathbf{n})\, d\sigma + \iiint\limits_{\Omega}\rho_s \mathbf{f} d\Omega = 0$$

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho E d\Omega + \iint\limits_{\sigma}\rho U_n I\, d\sigma + \iiint\limits_{\Omega}\rho_s(\mathbf{U}_s\mathbf{f} + q)\, d\Omega = 0$$

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho_s d\Omega + \iint\limits_{\sigma}\rho_s U_{sn} d\sigma = 0$$

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho_s \mathbf{U}_s d\Omega + \iint\limits_{\sigma}\rho_s U_{sn}\mathbf{U}_s\, d\sigma - \iiint\limits_{\Omega}\rho_s \mathbf{f} d\Omega = 0$$

$$\frac{d}{dt}\iiint\limits_{\Omega}\rho_s E_s d\Omega + \iint\limits_{\sigma}\rho_s U_{sn} E_s\, d\sigma - \iint\limits_{\Omega}\rho_s(\mathbf{U}_s\mathbf{f} + q)\, d\Omega = 0$$

$$E = e + U^2/_2, \quad I = E + p/\rho$$

where $\mathbf{f}$ is the specific force (per unit of particle mass) with which the gas acts on particles, q is the specific heat flow from gas to particles; $\mathbf{f}$ and q are assumed to be everywhere, particularly on the sheet, known functions of parameters of both media. Then (3.1) has a meaning for any (including discontinuous) parameter distribution, for instance, for a sheet for which the surface density is finite when ρ_s is infinite and, also, for lines called below "strings", along which it is the linear, not the surface density that is finite. We shall denote the surface and linear parameters of the sheet and string, respectively, by superscripts σ and l. Thus ρ_s^σ denotes the surface density per unit of the sheet surface, and ρ_s^l the linear density of particles per unit of string length.

Without going into the subject of tangential discontinuities of gas and particles, and on shock waves (in the gas), which was considered earlier, we draw the attention to the following. When the volume of particles is neglected, as assumed below, the parameters of gas at tangential discontinuities of particles are continuous, while the parameters of gas in transition through tangential discontinuities in gas and, also, shock waves become discontinuous [4, 5]. If in the last two cases $\mathbf{f}$ and q are finite, the parameters of particles retain their continuity over these discontinuities. In this case the explicit form of $\mathbf{f}$ and q is unimportant. It will become clear subsequently that the same requirements apply to the definition of $\mathbf{f}$ and q in the case of the string. The knowledge of expressions for $\mathbf{f}$ and q is generally necessary, although ever here there can be situations when a considerably lesser information may suffice.

The difference in the required information about $\mathbf{f}$ and q on the sheet and on other discontinuities is due to two factors. First, particles move along the sheet without intersecting it. Second, the density ρ_s on the sheet is inifinte (for finite ρ_s^σ)

and this results in finite changes of parameters of the gas flowing through the sheet. Hence explicit expressions for $\mathbf{f}$ and q are required for both, the definition of motion of the sheet itself and for the determination of gas parameter jumps on it. The density ρ_s in the string is so great (when ρ_s^σ is finite) that the flow of gas past it is like the flow past a thin hard filament and for finite $\mathbf{f}^l$ and q^l does not affect its motion, and since $\mathbf{f}^l$ and q^l are quantities defined per unit of particle mass, the assumption of their finiteness is reasonable.

In deriving the relationships on the sheet we shall limit the analysis to the steady flow. Let σ be an arbitrary small area of the sheet whose boundary is γ, and the unit vector of the normal to σ, i.e. $\mathbf{n}$ is oriented along the stream. By definition

$$\mathbf{n}\cdot\mathbf{U}_s^\sigma = 0 \tag{3.2}$$

Let $\mathbf{N}$ be the unit vector of the outer normal to γ and tangent to sheet $U_{sN}^\sigma = \mathbf{U}_s^\sigma\cdot\mathbf{N}$, and $\mathbf{U}_\tau = \mathbf{U} - U_n\mathbf{n}$ be the component of $\mathbf{U}$ tangent to the sheet. The relations on the stationary sheet are then by virtue of (3.1), of the form

$$\begin{gathered}[\rho U_n] = 0, \quad [p + \rho U_n^2] + \rho_s^\sigma f_n^\sigma = 0, \quad (\rho U_n)_-[\mathbf{U}_\tau] + \rho_s^\sigma \mathbf{f}_\tau^\sigma = 0 \\ (\rho U_n)_-[I] + \rho_s^\sigma(\mathbf{U}_s^\sigma\mathbf{f}^\sigma + q^\sigma) = 0 \\ \iint_\sigma [\rho_s U_{sn}]\,d\sigma + \oint_\gamma \rho_s^\sigma U_{sN}^\sigma d\gamma = 0, \quad \iint_\sigma ([\rho_s U_{sn}\mathbf{U}_s] - \\ \rho_s^\sigma\mathbf{f}^\sigma)\,d\sigma + \oint_\gamma \rho_s^\sigma U_{sN}^\sigma \mathbf{U}_s^\sigma\,d\gamma = 0, \quad \iint_\sigma \{[\rho_s U_{sn}E_s] - \\ \rho_s^\sigma(\mathbf{U}_s^\sigma\mathbf{f}^\sigma + q^\sigma)\}\,d\sigma + \oint_\gamma \rho_s^\sigma U_{sN}^\sigma E_s^\sigma d\gamma = 0\end{gathered} \tag{3.3}$$

where f_n^σ and $\mathbf{f}_\tau^\sigma$ are the respective components of $\mathbf{f}^\sigma$. Note that in this model according to the last of Eqs. (3.3) an exchange of kinetic and internal energies takes place in "the gas of particles" at the sheet, although there is no such exchange outside the latter.

If the vector $\mathbf{A}^\sigma$ tangent to the sheet has a continuous component A_N^σ normal to any curve γ on the sheet, then formula

$$\oint_\gamma A_N^\sigma d\gamma = \iint_\sigma \nabla_\sigma \mathbf{A}^\sigma d\sigma \quad \left(\nabla_\sigma\mathbf{A}^\sigma = \lim_{\sigma\to 0}\frac{1}{\sigma}\oint_\gamma A_N^\sigma\,d\gamma\right) \tag{3.4}$$

with the "divergence operator" ∇_σ on σ, is valid.

Let A_k be the projection of $\mathbf{A}$ on the k-axis of a Cartesin coordinate system. Then, when (3.4) holds, the last three of equalities (3.3) yield the equations in partial derivatives with two independent variables (e.g., curvilinear orthogonal coordinates on σ) which must be satisfied by the continuous surface parameters of particles on the sheet

$$\begin{gathered}\nabla_\sigma(\rho_s^\sigma\mathbf{U}_s^\sigma) + [\rho_s U_{sn}] = 0 \\ \nabla_\sigma(\rho_s^\sigma\mathbf{U}_s^\sigma U_{sk}^\sigma) + [\rho_s U_{sn}U_{sk}] - \rho_s^\sigma f_k^\sigma = 0 \\ \nabla_\sigma(\rho_s^\sigma\mathbf{U}_s^\sigma E_s^\sigma) + [\rho_s U_{sn}E_s] - \rho_s^\sigma(\mathbf{U}_s^\sigma\mathbf{f}^\sigma + q^\sigma) = 0\end{gathered} \tag{3.5}$$

The transition in (3.5) from ∇_σ to an operator defined in Cartesian or other coordinates is carried out in the conventional manner. Without doing this for the general case, we rewrite (3.5), as an example, for plane and an axisymmetric flows. In conformity with (3.2) and (3.5) the relationships

$$x' = u_s^\sigma / V_s^\sigma, \quad y' = v_s^\sigma / V_s^\sigma, \quad (y^\nu \rho_s^\sigma V_s^\sigma)' + y^\nu [\rho_s V_{sn}] = 0 \qquad (3.6)$$
$$(y^\nu \rho_s^\sigma V_s^\sigma u_s^\sigma)' + y^\nu ([\rho_s V_{sn} u_s] - \rho_s^\sigma f_x^\sigma) = 0$$
$$(y^\nu \rho_s^\sigma V_s^\sigma v_s^\sigma)' + y^\nu ([\rho_s V_{sn} v_s] - \rho_s^\sigma f_y^\sigma) = 0$$
$$(y^\nu \rho_s^\sigma V_s^\sigma \Gamma_s^\sigma)' + y^\nu ([\rho_s V_{sn} \Gamma_s] - \rho_s^\sigma y^\nu f_{z,\omega}^\sigma) = 0$$
$$(y^\nu \rho_s^\sigma V_s^\sigma E_s^\sigma)' + y^\nu \{[\rho_s V_{sn} E_s] - \rho_s^\sigma (\mathbf{U}_s^\sigma \mathbf{f}^\sigma + q^\sigma)\} = 0$$

are satisfied on the sheet besides the first four of finite equalities (3.3). In (3.6) the prime denotes the derivative $d/d\tau$, where τ is the distance measured along the line of intersection of the sheet with the xy-plane; xyz or $xy\omega$ are Cartesian or cylindrical coordinates; $\nu = 0$ and 1, respectively, in the plane and axisymmetric cases; indices x, y, z and ω denote projections of $\mathbf{f}^\sigma$ on respective axes; $\Gamma = y^\nu w$; projections of $\mathbf{U}$ on coordinate axes are denoted by u, v and w, and $\mathbf{V}$ is the "meridional" component of $\mathbf{U}$, and $V = \sqrt{u^2 + v^2}$.

In some cases, for instance when the sheet is formed in the plane of symmetry $U_{n-} = U_{n+} = 0$, the gas does not intersect it. Here the first of Eqs. (3.3) is automatically satisfied, the second yields a zero compression jump $[p] = 0$ because of $f_n^\sigma = 0$, and the third and fourth yield $\mathbf{f}_\tau^\sigma = 0$ and $q^\sigma = 0$. Since in the case of sheets of similar type $\mathbf{f}^\sigma$ and q^σ fall out from (3.3), (3.5), and (3.6), explicit expressions for $\mathbf{f}^\sigma$ and q^σ are unnecessary. As regards the particles whose streamlines lie on the sheet, we have, in accordance with the previously obtained equalities, $\mathbf{U}_\tau^\sigma = \mathbf{U}_{s\tau}^\sigma$ and $T^\sigma = T_s^\sigma$, where T is the temperature. Here and above it was assumed that $\mathbf{f} \sim \mathbf{U} - \mathbf{U}_s$ and $q \sim T - T_s$, hence when q^σ, $\mathbf{f}^\sigma$, or any component of $\mathbf{f}^\sigma$ indicates that the temperature and velocity of gas and particles or their components are the same.

When trajectories intersect "particles of the sheet", a discontinuity line, called above the string and characterized by the finite linear density ρ_s^l, is generated on the sheet. The relations that are satisfied along the steady string are readily obtained from the last three of Eqs. (3.3). Without adducing these, we would point out that along the string we have, owing to ρ_s^σ becoming infinitely great, the equalities $\mathbf{f}^l = 0$ and $q^l = 0$. Hence explicit expressions for $\mathbf{f}^l$ and q^l are generally not required, and the string itself is a "singular" streamline of the gas, which in no way affects the flow of gas outside the string. Sometimes the string may appear not on the sheet but in the region of the stream continuity. As an example, we point out the possibility of string formation on the axis of symmetry of an axisymmetric stream. The instant and the point or line of origin of the string or sheet are determined by conditions similar to those of shock wave generation in gasdynamics. As a rule, the surface or linear density of particles at the point or along the line of "origin" is zero. The exception is the focusing of particle trajectories (streamlines), and as well as the already considered in essence in Sect. 1 discontinuity formed on the line or surface of "input data".

4. Let us consider some of the singularities resulting from the introduction of the sheet in electro-gasdynamics. In the case of two-fluid model (gas and charged particles) to which the subsequent analysis is confined, these singularities are as follows. Let $\mathbf{E}$ be the electric field intensity, the gas permittivity be equal to that in vacuum, the magnetic field intensity be negligibly small, and g and g^σ be the volumes and surface densities of particle and sheet charges, respectively. If δ is the charge per unit mass of particles, then $g = \delta\rho_s$ and $g^\sigma = \delta\rho_s^\sigma$. For the components E_n and $\mathbf{E}_\tau$ of vector $\mathbf{E}$, respectively normal and tangent to the sheet, we have

$$[E_n] = \delta\rho_s^\sigma, \quad [\mathbf{E}_\tau] = 0$$

The equation of motion for the sheet must take into account all forces, including the electrical, external to the considered sheet element. The component of these forces tangent to the sheet is $\delta\rho_s^\sigma\,\mathbf{E}_\tau$. Projection of these forces on the normal to the sheet is calculated using the respective components of intensity of the field generated by charges "external" to the particular element. Neglecting the sheet thickness it possible to show that $(E_{n+} + E_{n-})/2$ defines that component and, consequently that the sought projection is $\delta\rho_s^\sigma\,(E_{n+} + E_{n-})/2$.

Without going into details, we note in conclusion the following. The appearance in the above investigation of the sheet and string is not much of a "physical phenomenon", as the result of the specific model (one-, two-fluid, etc.) of continuous medium. Hence one should not be surprised by the notion of ρ_s becoming infinitely great, i.e. exceeding the density of particle material, or that the sheet of electrically charged particles does not in this approximation scatter owing to repulsion of like charges. This aspect limits the value of the analysis of the sheet development. We shall, nevertheless, point out that in the two-fluid model with uncharged particles the sheet is evolutionary, when particles reach it from both sides and the normal velocity components of gas (in a coordinate system attached to the sheet) are either sub- or supersonic ahead of and behind the sheet. When these conditions are not satisfied an uncharged sheet becomes nonevolutionary.

The passing to a multi-fluid model with unlimited number of media is one of the devices for eliminating discontinuities of the sheet and string type. It is analogous to taking into consideration viscosity and heat conduction which blurr compression shocks in a perfect gas. One should however bear in mind that passing to more than two-fluid models not only complicates calculations but, also, requires additional information about the interaction between various media. In models which admit the sheet and string the required information is of a more "phenomenological" kind, and reduces to the specification of expressions for $\mathbf{f}^\sigma$ and q^σ. As previously indicated, it is possible in many cases to avoid these expressions. Moreover, the interaction between particles on intersecting trajectories reduces the width of the multi-fluid zones and, thus, reduces the errors associated with the substitution of discontinuities of the considered type for such zones. Finally, we would point out that the previously investigated in electro-gasdynamics discontinuities with a surface charge [7] differ from discontinuities of the sheet type.

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